Last time : linear functions $f : \mathbb{R}^n \to \mathbb{R}^m$ can be added and multiplied by constants

This is not surprising, since $\exists$ 1-to-1 correspondence

$$\{\text{linear functions } \mathbb{R}^n \to \mathbb{R}^m\} \longleftrightarrow \{m \times n \text{ matrices}\}$$

$$f(x) = Ax \quad , \forall x \in \mathbb{R}^n$$

and matrices can be added and multiplied by const

We also introduced the notion of injective, surjective and bijective functions

THM 6.5: assume $f(x) = Ax$ for some $A \in \mathbb{R}^{m \times n}$

(1) f is injective $\iff$ columns of A are linearly independent
$\iff$ REF(A) has n pivots $(m \geq n)$

(2) f is surjective $\iff$ columns of A span $\mathbb{R}^m$
$\iff$ REF(A) has m pivots $(m \leq n)$

(3) f is bijective $\iff$ columns of A are linearly independent and span $\mathbb{R}^m$
$\iff$ $m=n$ and REF(A) has $m=n$ pivots

Addition of linear functions corresponds to $\underbrace{\text{addition of matrices}}$

$$A, B \in \mathbb{R}^{m\times n} \rightsquigarrow (A+B)_{ij} = A_{ij} + B_{ij} \quad \forall\, 1\le i\le m,\ 1\le j\le n$$

$$f(x) = Ax$$
$$g(x) = Bx$$
$$\overline{}$$
$$(f+g)(x) = Sx$$

What is the relation between S and A, B?

$$A = (A_1 \ldots\ldots A_n) \quad , \quad A_i = f(e_i) \in \mathbb{R}^m \ni g(e_i) = B_i$$

$$B = (B_1 \ldots\ldots B_n) \qquad S_i = (f+g)(e_i) = f(e_i) + g(e_i) = A_i + B_i$$

$$S = (S_1 \ldots\ldots S_n) \qquad \text{where } e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

i-th spot

$$\Downarrow$$

$$S_1 = A_1 + B_1, \ldots\ldots, S_n = A_n + B_n$$

Example:

$$A = \begin{pmatrix} 3 & -2 \\ 7 & 3 \\ 4 & 9 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 3 \\ 7 \\ 4 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -2 \\ 3 \\ 9 \end{pmatrix}$$

$$B = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \qquad B_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad B_2 = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

(+)

$$S = \begin{pmatrix} 3+a & -2+d \\ 7+b & 3+e \\ 4+c & 9+f \end{pmatrix} \qquad S_1 = \begin{pmatrix} 3+a \\ 7+b \\ 4+c \end{pmatrix} \qquad S_2 = \begin{pmatrix} -2+d \\ 3+e \\ 9+f \end{pmatrix}$$

Multiplying a matrix by a number $\lambda \in \mathbb{R}$

$$\lambda \begin{pmatrix} 3 & 2 & 8 \\ 9 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 3\lambda & 2\lambda & 8\lambda \\ 9\lambda & 5\lambda & -\lambda \end{pmatrix}$$

Properties of addition of matrices $(\forall\, A, B, C \in \mathbb{R}^{m \times n})$

- $A + B = B + A$
- $A + (B + C) = (A + B) + C$
- $A + \underbrace{O}_{\text{zero matrix}} = A$

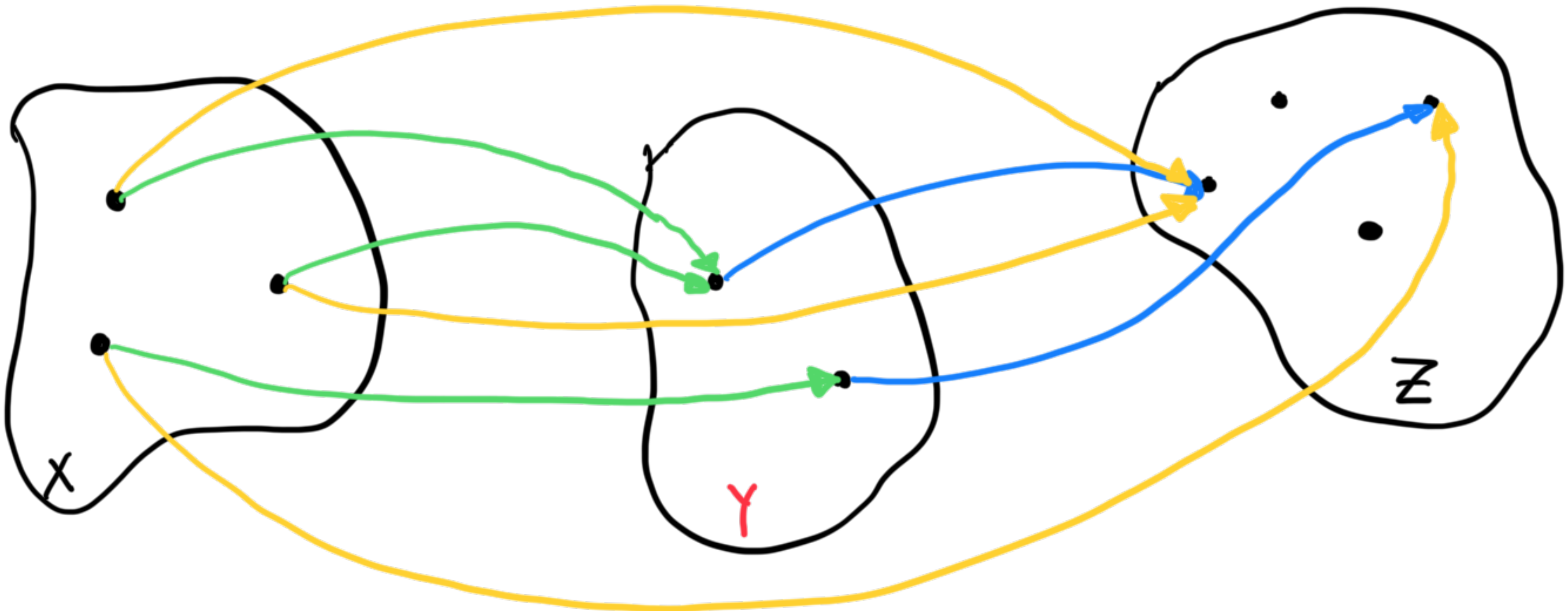
$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

- $\lambda (A + B) = \lambda A + \lambda B$
- $\lambda (\mu A) = (\lambda \mu) A$
- $(\lambda + \mu) A = \lambda A + \mu A$
- $\underbrace{0}_{\text{number}} A = \underbrace{O}_{\text{matrix}}$

- $$\begin{pmatrix} 3 & 7 \\ 5 & 9 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 2 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 5 & 11 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \\ 7 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 7 \\ 5 & 9 \\ -1 & 2 \end{pmatrix}$$

Composition of functions $f: Y \to Z$, $g: X \to Y$

$f \circ g$:



Composition is only defined when domain(f) = codomain(g)

DEF 7.1 : $f \circ g : X \longrightarrow Z$ is given by

composition

$$(f \circ g)(x) = f(g(x)) \in Z, \ \forall x \in X$$

Ex: $f : \mathbb{Z} \to \mathbb{Z}$ $\quad f(k) = 2k$
$g : \mathbb{Z} \to \mathbb{Z}$ $\quad g(k) = 3k$ $\quad , \forall k \in \mathbb{Z}$

$f \circ g : \mathbb{Z} \to \mathbb{Z}$ $\quad (f \circ g)(k) = f(g(k)) = f(3k) = 2 \cdot 3k = 6k$

Ex: $\forall$ set X, $Id_X : X \longrightarrow X$ is called the identity function
$x \longrightarrow x$

Then $\forall$ set Y, $\forall$ function $f : X \longrightarrow Y$, we have

$$f \circ Id_X = Id_Y \circ f = f \qquad (*)$$

$\forall x \in X$, let us see where each of these functions send x

$$f \circ Id_X(x) = f(Id_X(x)) = f(x)$$
$$Id_Y \circ f(x) = Id_Y(f(x)) = f(x)$$

These equalities imply $(*)$

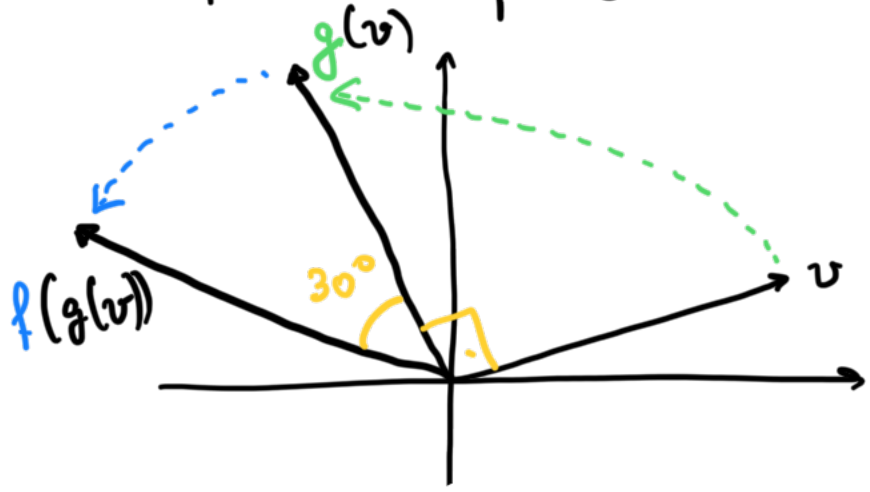
What about $\underbrace{Id_X \circ f}_{\text{not defined}}$ and $\underbrace{f \circ Id_Y}_{\text{not defined}}$? (unless $X = Y$)

Always, $f : Y \longrightarrow Z$, $g : X \longrightarrow Y$ compose only if domain (f) = codomain (g)

From now on, let's discuss compositions of linear functions

Ex: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is rotation by $90°$

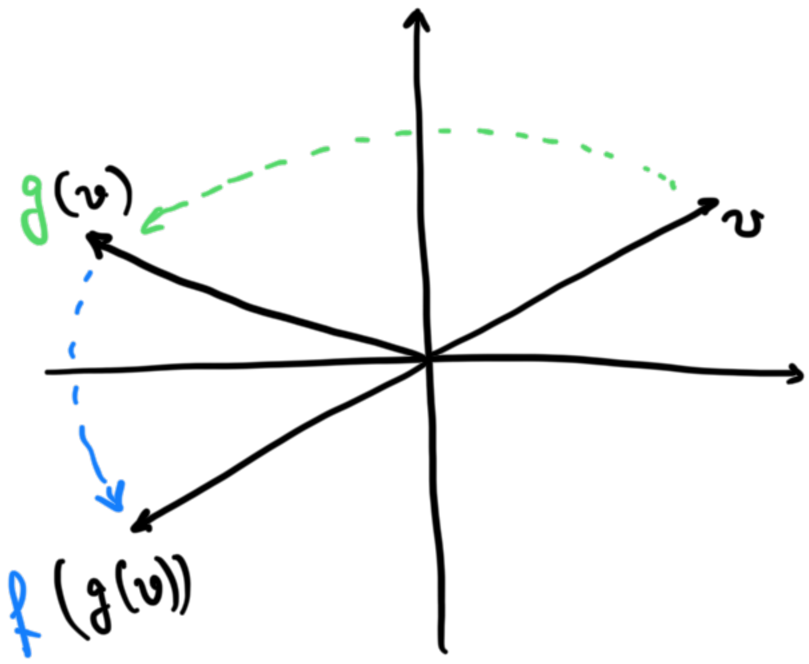
$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is rotation by $30°$



$f \circ g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is rotation by $120°$

Ex: f is reflection in —

g is reflection in |



$f \circ g$ is reflection in $\bullet$, i.e.

$(f \circ g)(v) = -v \ , \forall\, v \in \mathbb{R}^2$

Prop: if $\begin{array}{l} f : \mathbb{R}^m \longrightarrow \mathbb{R}^d \\ g : \mathbb{R}^n \longrightarrow \mathbb{R}^m \end{array}$ are linear

then $f \circ g : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ is also linear

Proof:

$$(f\circ g)(v+v') = f(g(v+v')) \overset{g \text{ linear}}{=} f(g(v)+g(v')) \overset{f \text{ linear}}{=} f(g(v)) + f(g(v')) = (f\circ g)(v) + (f\circ g)(v')$$

$$(f\circ g)(cv) = f(g(cv)) \underset{g \text{ linear}}{=} f(c\,g(v)) \underset{f \text{ linear}}{=} c\,f(g(v)) = c\,(f\circ g)(v)$$

Hence if $\begin{array}{l} f(x) = \overset{d\times m}{A}x \\ g(x) = \underset{m\times n}{B}x \end{array}$ then $(f\circ g)(x) = \underset{d\times n}{M}\,x$

How to get M from A and B?

The answer is matrix multiplication, which we now define

$$M = \left(M_1 \cdots\cdots M_n \right) \qquad \text{and } M_1, \ldots, M_n \in \mathbb{R}^d$$

$$M_i = (f \circ g)(e_i) = f(\underbrace{g(e_i)}_{B_i}) = f(B_i) = AB_i$$

Rule: M is the matrix whose columns are $AB_1, \ldots, AB_n$ where $B_1, \ldots, B_n$ are the columns of B

Ex: $A = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 0 & 1 \\ 9 & 0 & -3 \\ 5 & 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 5 \\ 3 & 0 \\ -1 & 7 \end{pmatrix} \implies M = AB = \begin{pmatrix} 1 & 16 \\ 1 & 12 \\ 21 & 24 \\ 16 & 25 \end{pmatrix}$

General rule for matrix multiplication

$$\underset{d \times m}{A} \; \underset{m \times n}{B} \in \mathbb{R}^{d \times n}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \end{pmatrix}$$

$$\begin{pmatrix} \cdots & a_{22} & \cdots & a_{2m} \\ & \vdots & & \\ a_{d1} & a_{d2} & \cdots & a_{dm} \end{pmatrix} \begin{pmatrix} b_{21} & \cdots & b_{2n} \\ \vdots & & \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}b_{11}+\dots+a_{1m}b_{m1} & \cdots\cdots & a_{11}b_{1n}+\dots+a_{1m}b_{mn} \\ a_{21}b_{11}+\dots+a_{2m}b_{m1} & & a_{21}b_{1n}+\dots+a_{2m}b_{mn} \\ \vdots & & \vdots \\ a_{d1}b_{11}+\dots+a_{dm}b_{m1} & \cdots\cdots & a_{d1}b_{1n}+\dots+a_{dm}b_{mn} \end{pmatrix}$$

Closed formula for matrix multiplication:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

THM 7.2: composition of linear functions corresponds to multiplication of matrices

i.e. if $f(x) = Ax$ and $g(x) = Bx$ then $(f \circ g)(x) = (AB)x$

matrix multiplication

Ex: $A = \begin{pmatrix} 2 & 3 & 9 & 7 \\ -1 & 0 & 1 & 5 \\ 0 & 8 & -2 & 4 \end{pmatrix}$ $\quad B = \begin{pmatrix} x & a \\ y & b \\ z & c \end{pmatrix}$

4×2

3×4 $\qquad t \quad d$

$$AB = \left(\quad \right) = \left(\begin{array}{c|c} 2x + 3y + 9z + 7t & 2a + 3b + 9c + 7d \\ \hline -x \qquad + z + 5t & -a \qquad + c \quad + 5d \\ \hline 8y - 2z + 4t & 8b - 2c + 4d \end{array} \right)$$

3×2

where $\quad = A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2x + 3y + 9z + 7t \\ -x \qquad + z + 5t \\ 8y - 2z + 4t \end{pmatrix}$

$\quad = A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2a + 3b + 9c + 7d \\ -a \qquad + c + 5d \\ 8b - 2c + 4d \end{pmatrix}$

Properties of matrix multiplication

- $A(BC) = (AB)C$ — associativity
- $A(B+C) = AB + AC$
- $(B+C)A = BA + CA$ } distributivity
- $\lambda(AB) = (\lambda A)B = A(\lambda B)$

✡ $\underset{m\times m}{I_m} \cdot \underset{m\times n}{A} = \underset{m\times n}{A} \cdot \underset{n\times n}{I_n} = \underset{m\times n}{A}$ } identity property

where $I_n = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$ is the n×n identity matrix (square)

Proof of ✱ :

$$\underset{I_4}{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}}\underset{A}{\begin{pmatrix}a_{11}&a_{12}&\cdots\\a_{21}&a_{22}&\cdots\\a_{31}&a_{32}&\cdots\\a_{41}&a_{42}&\cdots\end{pmatrix}} = \underset{A}{\begin{pmatrix}a_{11}&a_{12}&\cdots\\a_{21}&a_{22}&\cdots\\a_{31}&a_{32}&\cdots\\a_{41}&a_{42}&\cdots\end{pmatrix}}$$

$$\underset{A}{\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\\vdots&\vdots&\vdots\end{pmatrix}}\underset{I_3}{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}} = \underset{A}{\begin{pmatrix}\overset{a_{11}}{a_{11}\cdot 1+a_{12}\cdot 0+a_{13}\cdot 0}&\overset{a_{12}}{a_{11}\cdot 0+a_{12}\cdot 1+a_{13}\cdot 0}&\overset{a_{13}}{a_{11}\cdot 0+a_{12}0+a_{13}\cdot 1}\\\overset{a_{21}}{a_{21}\cdot 1+a_{22}0+a_{23}0}&\overset{a_{22}}{a_{21}0+a_{22}1+a_{23}0}&\overset{a_{23}}{a_{21}0+a_{22}0+a_{23}\cdot 1}\\\vdots&\vdots&\vdots\end{pmatrix}}$$

Alternative proof of ✱

$$A \rightsquigarrow f(x) = Ax \quad : \mathbb{R}^n \to \mathbb{R}^m$$

$$I_n \rightsquigarrow Id_{\mathbb{R}^n}(x) = x \quad : \mathbb{R}^n \to \mathbb{R}^n$$

identity function

$$I_n = \begin{pmatrix}1&0&\cdots&0&0\\0&1&\cdots&0&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&&1&0\\0&0&&0&1\end{pmatrix} \leftsquigarrow Id_{\mathbb{R}^n}(e_1) = e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \;\cdots\; Id_{\mathbb{R}^n}(e_n) = e_n = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

$$I_m A = A \cdot I_n = A \quad \text{for an } m \times n \text{ matrix } A$$

$$Id_{\mathbb{R}^m} \circ f = f \circ Id_{\mathbb{R}^n} = f$$ (we already proved this in ⊛)

Proof of $\underset{m\times n}{A}(\underset{n\times d}{B}\ \underset{d\times e}{C}) = (\underset{m\times n}{A}\ \underset{n\times d}{B})\underset{d\times e}{C}$

$f(x) = Ax \qquad f: \mathbb{R}^n \to \mathbb{R}^m$

$g(x) = Bx \qquad g: \mathbb{R}^d \to \mathbb{R}^n$

$h(x) = Cx \qquad h: \mathbb{R}^e \to \mathbb{R}^d$

$$f \circ (g \circ h) = (f \circ g) \circ h$$

$\forall x \in \mathbb{R}^e$,

$$LHS(x) = f \circ (g \circ h)(x) = f((g \circ h)(x)) = f(g(h(x)))$$

$$RHS(x) = (f \circ g) \circ h(x) = (f \circ g)(h(x)) = f(g(h(x)))$$

Non-properties of matrix multiplication:

- $AB \neq BA$ not commutative for general A, B

$$\begin{pmatrix} 7 & 5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 63 & 15 \\ -9 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 63 & 45 \\ -3 & 6 \end{pmatrix}$$

- $AB = O$ even if $A \neq O \neq B$

  O matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O.$$

- $AB = AC \not\Rightarrow B = C$ even if $A \neq O$ matrix
- $BA = CA \not\Rightarrow$